On a linear optical implementation of non local product states and on their indistinguishability

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In a recent paper Bennett et al. [1] have shown the existence of a basis of product states of a bipartite system with manifest non-local properties. In particular these states cannot be completely discriminated by means of bilocal measurements. In this paper we propose an optical realization of these states and we will show that they cannot be completely discriminate by means of a global measurement using only optical linear elements, conditional transformation and auxiliary photons.

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I. INTRODUCTION

Quantum optical systems are ideal for the experimental test of the foundation of quantum mechanics [2] as well as for the experimental implementation of quantum information protocols like quantum cryptography [3], quantum teleportation [4], quantum dense coding [5] and quantum computation [6]. In most of the above experiments the key point is the generation and the detection of entangled states. While the generation of various kind of entangled states is now part of the daily routine of a good laboratory the detection can be a surprisingly difficult task. The most typical example is probably the detection of Bell states [7], for which it has been shown the impossibility to build a setup able to discriminate with 100% efficiency all the four Bell states using only linear optical devices [8, 9, 10]. Such impossibility to discriminate the states of an orthogonal basis is by no means restricted to entangled systems. We will show that this difficulty is present also in the case of an orthogonal basis of a bipartite system which has been introduced in connection with non locality without entanglement. Non locality has always been associated with quantum entanglement. In a recent article however [1] Bennett et al have provided a counterexample by showing the existence of an orthogonal set of states of a bipartite system which, although not entangled, are not distinguishable by means of bilocal measurements. Given two particles, each of which described by a three dimensional Hilbert space, they construct the following orthogonal basis:

$$\begin{aligned}
|\psi_0\rangle &= |2\rangle_A \otimes |2\rangle \\
|\psi_{\pm 1}\rangle &= \frac{1}{\sqrt{2}} |1\rangle_A \otimes (|1\rangle \pm |2\rangle)_B \\
|\psi_{\pm 2}\rangle &= \frac{1}{\sqrt{2}} |3\rangle_A \otimes (|2\rangle \pm |3\rangle)_B \\
|\psi_{\pm 3}\rangle &= \frac{1}{\sqrt{2}} (|2\rangle \pm |3\rangle)_A \otimes |1\rangle_B \\
|\psi_{\pm 4}\rangle &= \frac{1}{\sqrt{2}} (|1\rangle \pm |2\rangle)_A \otimes |3\rangle_B
\end{aligned} \tag{1}$$

where A,B label the two particles and $|1\rangle\,|2\rangle\,|3\rangle$ are three orthogonal states for each particle.

The peculiar property of states (1) is that they cannot be reliably distinguished by two separate observers by means of any sequence of local operations even if they are allowed to exchange classical communication.

In this paper we propose an optical realization of states (1) and investigate the possibility to fully discriminate them with a *global measurement by means of linear elements*. A related problem has been investigated in connection with the possibility to discriminate Bell states. It has been shown [8, 9, 10]that it is not possible to perform a complete Bell measurement on a product Hilbert space of two two-level bosonic systems states by means of purely linear optical elements. One might expect that this is due to the entangled nature of the Bell states. However, following the line of [9], we will show that also states (1) are not fully distinguishable by a global measurement using only linear elements, even though they are not entangled.

II. THE SETUP

In our optical setup the three dimensional Hilbert space of each subsystem is mapped into the single photon state of three different modes of the electromagnetic field.

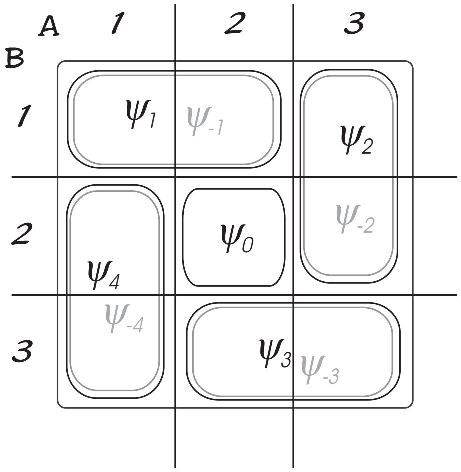

FIG. 1: Graphical representation of the states of ref. [1] as a system of dominos. The fact that even if these states are globally orthogonal their parts are not, is evident in the picture, where the measurement are represented as a cut along dashed lines.

The basis states $|1\rangle\,|2\rangle\,|3\rangle$ for each of the two subsystems will therefore be of the form $|i\rangle_A = a_i^\dagger\,|0\rangle\,, |i\rangle_B = b_i^\dagger\,|0\rangle$ where $a_i^\dagger, b_i^\dagger$ (1=1,2,3) are bosonic creation operators of three orthogonal modes and $|0\rangle$ is the vacuum state. In this notation states (1) are written as follows

$$\begin{aligned}
|\psi_0\rangle &= \hat{a}_2^\dagger \hat{b}_2^\dagger\,|0\rangle \\
|\psi_{\pm 1}\rangle &= \frac{1}{\sqrt{2}}\hat{a}_1^\dagger(\hat{b}_1^\dagger \pm \hat{b}_2^\dagger)\,|0\rangle \\
|\psi_{\pm 2}\rangle &= \frac{1}{\sqrt{2}}\hat{b}_1^\dagger(\hat{a}_3^\dagger \pm \hat{a}_2^\dagger)\,|0\rangle \\
|\psi_{\pm 3}\rangle &= \frac{1}{\sqrt{2}}\hat{a}_3^\dagger(\hat{b}_3^\dagger \pm \hat{b}_2^\dagger)\,|0\rangle \\
|\psi_{\pm 4}\rangle &= \frac{1}{\sqrt{2}}\hat{b}_3^\dagger(\hat{a}_1^\dagger \pm \hat{a}_2^\dagger)\,|0\rangle \qquad (2)
\end{aligned}$$

The impossibility to distinguish states (2) by means of bilocal measurements implies that they are not distinguishable by measuring directly the photon number of each individual mode. A first attempt to implement a collective measurement could be to mix the modes by means of linear devices and than to measure the output modes of such device. However, following [9] we will adopt a more general strategy. We will assume to have at our disposal a set of as many additional modes as we like, here indicated with bosonic creation operators $c_j^\dagger$, with any number of photons we like and we will assume that these auxiliary modes can be mixed with modes $a_i^\dagger, b_k^\dagger$ in a black box.

The output modes of this box are linked to the input ones by a unitary transformation U. It has been shown [11] that any such unitary transformations of modes can be obtained by means of linear optical devices, like beam splitters and phase shifters. To ensure the largest possible generality in our measurement apparatus we will assume the possibility to perform conditional measurements. In practice this means what follows: assume that a measurement is made on one selected output mode while the other are kept in a delay loop and that, according to the outcome of the measurement, these modes are fed into a selected further black box, in a cascade setup (see figure 2). The final assumption we will make is that our detectors have the ability to discriminate the number of incident photons. This assumption is clearly unrealistic. We will show, however, that even if such detectors were available, the measurement setup described above cannot discriminate states (2).

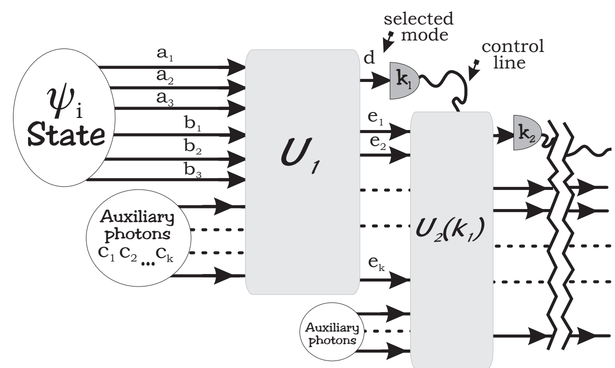

FIG. 2: Cascade setup in which the modes of the states (2) are mixed in a first "box" with auxiliary modes. Selected output mode is then measured and depending on its outcome the remaining output modes are fed in a new box. The process can be repeated over and over again

III. SYMMETRY PROPERTIES

In this section we will describe some symmetry properties of states (2) which are not only interesting per se but also will turn out useful in the following.

Consider the following transformation $\hat{\mathbf{T}}$ which permutes the modes of photon A with the ones of photon B:

$$\hat{\mathbf{T}} : \begin{cases} |i\rangle_A \;\rightarrow\; |i\rangle_B \\ |i\rangle_B \;\rightarrow\; |4-i\rangle_A \end{cases}$$

This is obviously a linear transformation. In the basis states $|1\rangle_A$,$|2\rangle_A$,$|3\rangle_A$,$|1\rangle_B$,$|2\rangle_B$,$|3\rangle_B$ $\hat{\mathbf{T}}$ takes the following matrix form

$$\hat{\mathbf{T}} = \begin{pmatrix} 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \\ 0&0&0&1&0&0 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \end{pmatrix}$$

The set of states (2) is globally invariant under the action of $\hat{\mathbf{T}}$ since

$$\begin{cases} |\psi_0\rangle & \xrightarrow{\hat{\mathbf{T}}} & |\psi_0\rangle \\ |\psi_{\pm 1}\rangle & \xrightarrow{\hat{\mathbf{T}}} & |\psi_{\pm 2}\rangle \\ |\psi_{\pm 2}\rangle & \xrightarrow{\hat{\mathbf{T}}} & |\psi_{\pm 3}\rangle \\ |\psi_{\pm 3}\rangle & \xrightarrow{\hat{\mathbf{T}}} & |\psi_{\pm 4}\rangle \\ |\psi_{\pm 4}\rangle & \xrightarrow{\hat{\mathbf{T}}} & |\psi_{\pm 1}\rangle \end{cases}$$

furthermore it is straightforward to verify that $\hat{\mathbf{T}}^4 = \hat{\mathbf{1}}$. Another linear transformation we will use in the following is the one which introduces a phase change of π on states $|2\rangle_A$ and $|2\rangle_B$ leaving unaltered all the others. In matrix form

$$\hat{\mathbf{S}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The action of $\hat{\mathbf{S}}$ on states (2) is simply

$$\hat{\mathbf{S}} : |\psi_{\mathbf{i}}\rangle \rightarrow |\psi_{-\mathbf{i}}\rangle$$

With $\hat{\mathbf{S}}, \hat{\mathbf{T}}$ form a group which leaves states (2) invariant. Furthermore, by repeated action of $\hat{\mathbf{S}}$ and $\hat{\mathbf{T}}$, it is possible to transform any $|\psi_i\rangle$ into any other $|\psi_j\rangle$, with the exception of $|\psi_0\rangle$ which is mapped onto itself. For instance we can transform ψ_1 into a generic $\psi_{\pm k}$ (with k=1...4) by acting with the operator

$$\hat{\mathbf{R}}_{\pm\mathbf{k}} = \hat{\mathbf{S}}^{\frac{1\pm 1}{2}} \cdot \hat{\mathbf{T}}^{k-1}$$

This implies that the problem of how to generate the states (2) reduces to the problem of how to generate one of them as the others can be obtained by repeated action of $\hat{\mathbf{S}}$ and $\hat{\mathbf{T}}$ and, as we have said already, this can be achieved by linear optical devices.

IV. AUXILIARY PHOTONS DO NOT INCREASE DISTINGUISHABILITY

We will now show that the use of auxiliary photons in the measurement setup described in section II does not help in increasing the distinguishability of states (2). The argument is a generalization to our more complex set of states of the one used in [9] in connection with the problem of distinguishing Bell states with an analogous setup. In this section we will outline the proof, leaving the details to appendix A.

As described already our measuring apparatus consists of a cascade of "black boxes", in which modes are linearly mixed, and partial measurements, which determine the sequence of unitary mixing. The first of such black box, denoted by U_1, is made out of linear optical elements and its input and output are a set of bosonic modes. The joint input modes consist of our six "system" modes $a_i^\dagger, b_k^\dagger$ and an arbitrary number of auxiliary modes $c_i^\dagger$. These input modes are unitarily mixed in the box into a set of output modes $e_i^\dagger, d^\dagger$ where the $d^\dagger$ mode is the one on which a measurement will be performed. The measurement outcome will determine the specific unitary mixing that will be performed in next step of the measurement, consisting of a second box U_2. While the measurement on mode $d^\dagger$ is performed the photons in the remaining $e_i^\dagger$ modes are kept in a waiting loop. The whole measurement procedure consists of a cascade of conditional measurements as described above.

Let's now look more in detail at the first block of the apparatus. The input state of U_1 can be written as

$$\left|\psi_i^{tot}\right\rangle = |\psi_{aux}\rangle \otimes |\psi_i\rangle = P_{aux}(c_k^\dagger) P_i(a_n^\dagger, b_m^\dagger)\,|0\rangle \qquad (3)$$

where $P_i(a_i^\dagger, b_i^\dagger)$ is a polynomial of degree 2 and $P_{aux}(c_k^\dagger)$ is a polynomial of arbitrary degree in the $c_k^\dagger$

The corresponding output state is

$$\left|\psi_i^{tot}\right\rangle = \tilde{P}_{aux}(d^\dagger, e_k^\dagger) \tilde{P}_{\psi_i}(d^\dagger, e_k^\dagger)\,|0\rangle \qquad (4)$$

Where $\tilde{P}_{aux}(d^\dagger, e_k^\dagger)$ and $\tilde{P}_{\psi_i}(d^\dagger, e_k^\dagger)$ are nothing but $P_{aux}(c_k^\dagger)$ e $P_i(a_n^\dagger, b_m^\dagger)\,|0\rangle$ written in terms of the creation and annihilation operators at the output of U_1.

We can expand $\tilde{P}_{aux}, \tilde{P}_{\Psi_i}$ in terms of decreasing powers of $d^\dagger$ as follows

$$\tilde{P}_{aux}(d^\dagger, e_k^\dagger) = \sum_{n=0}^{n_a} (d^\dagger)^n \tilde{Q}_a^{(n)}(e_k^\dagger) \qquad (5)$$

$$\tilde{P}_{\psi_i}(d^\dagger, e_k^\dagger) = \sum_{n=0}^{n_s} (d^\dagger)^n \tilde{Q}_{\psi_i}^{(n)}(e_k^\dagger) \qquad (6)$$

In (6) n_s is the largest order in $d^\dagger$ for the nine $\tilde{P}_{\Psi_i}$ and by definition is independent on index i ($\tilde{Q}_{\Psi_i}$ can be zero for some i). Analogously n_a is defined as the order in $d^\dagger$ of polynomial $\tilde{P}_{aux}$. We can therefore rewrite (4) as

$$\left|\psi_i^{tot}\right\rangle = \sum_{n,m=0}^{n_a,n_s} (d^\dagger)^{n+m} \tilde{Q}_a^{(n)}(e_n^\dagger) \tilde{Q}_{\psi_i}^{(m)}(e_k^\dagger)\,|0\rangle \qquad (7)$$

Out of the possible outcomes of the measurement of the number N of photons in mode d we will concentrate on two particular outcomes, namely those resulting in the highest number, N_{max} and $N_{max} - 1$ where $N_{max} = n_s + n_a$. The reason of this particular choice will be shortly evident.

Let's suppose now that the number of photons on the selected mode d is measured. If N is the outcome of such measurement the (unnormalised) conditional state of the remaining modes can be we written as

$$\left|\psi_i^{cond \to N}\right\rangle = \sum_{n=max\{0,N-n_s\}}^{min\{n_a,N\}} \tilde{Q}_a^{(n)} \tilde{Q}_{\psi_i}^{(N-n)} |0\rangle \tag{8}$$

If the input states are to be distinguishable the conditional states $\left|\psi_i^N\right\rangle$ *must be orthogonal for each possible value of* N, i.e.

$$\left\langle \psi_i^N \middle| \psi_j^N \right\rangle = 0 \qquad \forall N, i \neq j$$

In appendix A we will show that the two conditions

$$\begin{cases} \left\langle \psi_i^{N_{max}} \middle| \psi_j^{N_{max}} \right\rangle = 0 \\ \left\langle \psi_i^{N_{max}-1} \middle| \psi_j^{N_{max}-1} \right\rangle = 0 \end{cases} \tag{9}$$

can be simultaneously satisfied if and only if the two conditions

$$\begin{cases} \langle 0| \tilde{Q}_{\psi_i}^{(n_s)\dagger} \tilde{Q}_{\psi_j}^{(n_s)} |0\rangle = 0 \\ \langle 0| \tilde{Q}_{\psi_i}^{(n_s-1)\dagger} \tilde{Q}_{\psi_j}^{(n_s-1)} |0\rangle = 0 \quad (\text{for } n_s \neq 0) \end{cases} \tag{10}$$

are simultaneously satisfied. The important point is that (10) *do not depend on the auxiliary input states.* It is easy to convince oneself that this is the case since from (8) follows that

$$\langle 0| \tilde{Q}_{\psi_i}^{(N)\dagger} \tilde{Q}_{\psi_j}^{(N)} |0\rangle \propto \left\langle \psi_i^N \middle| \psi_j^N \right\rangle_{n_a=0} \tag{11}$$

where $\left|\psi_i^N\right\rangle_{n_a=0}$ is the conditional output state obtained from ψ_i in the absence of auxiliary photons when N photons are measured in mode d.

The central point of this section is that the fact that condition (9) implies condition (10) is equivalent to say that any pair of states ψ_i, ψ_j are distinguishable in the presence of auxiliary photons only if they are distinguishable in the absence of auxiliary photons. In other words auxiliary photons do not improve complete distinguishability.

V. IT IS IMPOSSIBLE TO BUILD A COMPLETE LINEAR DISCRIMINATOR

We will now show that it is impossible for states (2) to satisfy

$$\left\langle \psi_i^{n_s} \middle| \psi_j^{n_s} \right\rangle_{n_a=0} = 0 \tag{12a}$$

$$\left\langle \psi_i^{n_s-1} \middle| \psi_j^{n_s-1} \right\rangle_{n_a=0} = 0 \quad (\text{for } n_s \neq 0) \tag{12b}$$

for all i, $j \in \{-4..4\}$ ($i \neq j$).

In the absence of auxiliary photons states $|\psi_i\rangle$ can be written in terms of a polynomial of creation operators as

$$|\psi_i\rangle = P_{\psi_i}\left(a_1^\dagger, a_2^\dagger, a_3^\dagger, b_1^\dagger, b_2^\dagger, b_3^\dagger\right)|0\rangle$$

Let's now define the creation operator vector as

$$\mathbf{A} \equiv \left(\hat{a}_1^\dagger, \hat{a}_2^\dagger, \hat{a}_3^\dagger, \hat{b}_1^\dagger, \hat{b}_2^\dagger, \hat{b}_3^\dagger, \{c_k^\dagger\}\right)^T$$

where the $\{c_k^\dagger\}$ are a possible set of (empty) auxiliary modes. Since the ψ_i are two photon states they can be written in terms of a real symmetric matrix $\mathbf{M}^{(i)}$ as follows

$$|\psi_i\rangle = \mathbf{A}^T \mathbf{M}^{(i)} \mathbf{A} |0\rangle$$

where the exact form of $\mathbf{M}^{(i)}$ can be obtained from (2).

If $\mathbf{U}$ is a generic unitary matrix transforming the input modes into the output ones of our apparatus than

$$|\psi_i\rangle = \tilde{\mathbf{A}}^T \tilde{\mathbf{M}}^{(i)} \tilde{\mathbf{A}} |0\rangle \tag{13}$$

where

$$\tilde{\mathbf{M}}^{(i)} = \mathbf{U}^T \mathbf{M}^{(i)} \mathbf{U}$$

and

$$\tilde{\mathbf{A}} = \mathbf{U}^\dagger \mathbf{A} = \left(d^\dagger, e_1^\dagger, e_2^\dagger, \dots\right)^T$$

with $d^\dagger$ corresponding to the detected output mode.

States (13) can than be written as

$$\begin{aligned} |\psi_i\rangle = {} & \tilde{M}_{00}^{(i)} (d^\dagger)^2 |0\rangle + \\ & + 2\sum_{k=1}^{D} \tilde{M}_{0k}^{(i)} d^\dagger e_k^\dagger |0\rangle + \sum_{k,l=1}^{D} \tilde{M}_{kl}^{(i)} e_k^\dagger e_l^\dagger |0\rangle \end{aligned} \tag{14}$$

where $\tilde{M}_{kl}^{(i)}$ is the generic matrix element of $\tilde{\mathbf{M}}^{(i)}$ whose dimension $D+1$ corresponds to the number of output modes involved.

Let's write $\mathbf{U}$ as

$$\mathbf{U} = \begin{pmatrix} u_0 & \underline{\mathrm{r}}_0 \\ u_1 & \underline{\mathrm{r}}_1 \\ \vdots & \vdots \\ u_D & \underline{\mathrm{r}}_D \end{pmatrix}$$

where u_i are the element of the first column of the matrix and $\underline{\mathrm{r}}_i$ (with $i \in \{0..D\}$) are vectors of dimension D representing the remaining elements of row i^{th}. As a consequence of the unitarity of $\mathbf{U}$ we have

$$u_i^* u_j + \underline{\mathrm{r}}_i^\dagger \cdot \underline{\mathrm{r}}_j = \delta_{ij} \tag{15}$$

We define the columns vector $\underline{c}_{\mathbf{0}}$ whose elements are the first 6 elements of the 0^{th} column of $\mathbf{U}$:

$$\underline{c}_{\mathbf{0}} = (u_0, \ldots, u_5, 0, \ldots, 0)^T \qquad (16)$$

We recall that n_s is the highest degree of $d^\dagger$ in polynomials $\tilde{\mathbf{A}}^T \tilde{\mathbf{M}^{(i)}} \tilde{\mathbf{A}}$ for all values of i, in other words the maximum number of photons which can be detected in d for all possible input states $\{\psi_i,\, i \in \{-4\ldots4\}\}$. Obviously n_s can assume only values $0, 1, 2$, depending on the specific choice of $\mathbf{U}$ and d. We will now show that for all possible value of n_s it is impossible to satisfy simultaneously (12a) and (12b).

$n_s = 0$: this corresponds to a bad choice of mode d, as the monitored mode would be decoupled from the input ones for all possible input state.

$n_s = 1$: this corresponds to $\tilde{M}^{(i)}_{00} = 0$ for all value of i (see (14)). This implies that

$$\begin{aligned} \tilde{M}^{(i)}_{00} &= \sum_{k,l=0}^{D} M^{(i)}_{kl} u^*_k u_l = \\ &= \underline{c}_{\mathbf{0}}{}^T \cdot \mathbf{M}^{(i)} \cdot \underline{c}_{\mathbf{0}} = 0 \end{aligned} \qquad \forall\, i \in \{-4..4\} \qquad (17)$$

The above relation is a constrain on $\underline{c}_{\mathbf{0}}$ which we will now show to be incompatible with (12a).

To this end we note that from (14) follows that after the detection of one photon in mode d the remaining modes are left in the (unormalised) conditional state

$$\begin{aligned} \left|\psi_i^{cond\to1}\right\rangle &= \sum_{k=1}^{D} \tilde{M}^{(i)}_{0k} e^\dagger_k \left|0\right\rangle \\ &= \vec{\boldsymbol{M}}^{(i)}_0 \cdot \tilde{\mathbf{A}} \left|0\right\rangle \end{aligned} \qquad (18)$$

where for convenience of notation we have introduced the vector

$$\vec{\boldsymbol{M}}^{(i)}_0 = \sum_{k,l=1}^{5} M^{(i)}_{kl} u_k \underline{r}_l = (0, \tilde{M}^{(i)}_{01}, \tilde{M}^{(i)}_{02}, \ldots, \tilde{M}^{(i)}_{0D}) \quad (19)$$

From (18) and (19) follows that the trivial solution $\underline{c}_{\mathbf{0}} = \underline{0}$ implies $\left|\psi_i^{cond\to1}\right\rangle = 0\ \forall i$, i.e. $n_s = 0$. We must therefore look for possible nontrivial solutions of eq. (17) compatible with (12a), which in this particular case reads

$$\left\langle\psi_i^{cond\to1}\right| \left.\psi_j^{cond\to1}\right\rangle = \vec{\boldsymbol{M}}^{(i)\dagger}_0 \cdot \vec{\boldsymbol{M}}^{(j)}_0 = 0 \qquad (20)$$

However, as shown in appendix B conditions (17) and (20) are compatible only with the trivial solution. This implies that it is not possible to build a complete discriminator for $n_s = 1$.

$n_s = 2$ this corresponds a non zero probability to measure two photons in mode d for some ψ_i which implies $\tilde{M}^{(i)}_{00} \neq 0$ for at least one value of i. On the other hand condition (12a) can be satisfied in this specific case if and only if $\tilde{M}^{(i)}_{00} \neq 0$ for at most one value of i, which we will denote by i_o. Condition (12a) than becomes

$$\tilde{M}^{(i)}_{00} = \underline{c}_{\mathbf{0}}{}^T \cdot \mathbf{M}^{(i)} \cdot \underline{c}_{\mathbf{0}} = 0 \qquad i \neq i_o \qquad (21)$$

and (12b) becomes equivalent to condition (20). In order to complete our proof it will therefore be enough to show that whatever the value of i_o conditions (21) and (20) cannot be simultaneously satisfied. Suppose in particular that they are not satisfied for $i_o = 1$, the symmetry analysis carried out in the previous section immediately lead to the conclusion that they cannot be satisfied by any other value i_o (apart from $i_o = 0$). We have shown that it is always possible to build a linear operator $\hat{\mathbf{R}}_{\mathbf{i}}$ which transforms ψ_1 into ψ_i ($i \neq 0$) and leaves the set of states $\{\psi_i\}$ globally invariant. If there were a linear operator $\mathbf{U}'$ such to satisfy conditions (21) and (20) for any value of $i_o \neq 0$ than $\mathbf{U} = \hat{\mathbf{R}}^\star_{\mathbf{i}} \cdot \mathbf{U}' \cdot \hat{\mathbf{R}}_{\mathbf{i}}$ would satisfy the same conditions for $i_o = 1$, which contradicts our initial assumption. The problem than reduces to the analysis of the cases $i_o = 0$ and $i_o = 1$. Such analysis, straightforward but tedious (see appendix B), leads to the result that indeed for both values of i_o conditions (21) and (20) are incompatible.

VI. CONCLUSIONS

In this paper we have proposed an optical realization of states (1). Bennett et al. [1] have shown that they cannot be discriminated by means of local action and classical communication. We have demonstrated that to add the possibility of global interference it is still not sufficient. In other words we have shown the impossibility to fully discriminate them by means of a global measurement using linear elements, like beam splitters and phase shifters, delay lines and electronically switched linear elements, photodetectors and auxiliary photons.

The impossibility to implement such a measurement has already been shown for the set of maximally entangled Bell states. We have proved an analogous no-go theorem for a set of states which, although non local, are not entangled. This opens new questions on which class of photon states can be in general be fully discriminated by means of linear optical systems.

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APPENDIX A

In this appendix we will show that the two conditions

$$\begin{cases} \left\langle \psi_i^{N_{max}} \middle| \psi_j^{N_{max}} \right\rangle = 0 \\ \left\langle \psi_i^{N_{max}-1} \middle| \psi_j^{N_{max}-1} \right\rangle = 0 \end{cases} \tag{A1a}$$

can be simultaneously satisfied if and only if the following conditions

$$\begin{cases} \langle 0| \tilde{Q}_{\psi_i}^{(n_s)\dagger} \tilde{Q}_{\psi_j}^{(n_s)} |0\rangle = 0 \\ \langle 0| \tilde{Q}_{\psi_i}^{(n_s-1)\dagger} \tilde{Q}_{\psi_j}^{(n_s-1)} |0\rangle = 0 \end{cases} \tag{A1b}$$

are simultaneously satisfied. From (8) follows that the scalar product between the (unormalised) states $\left|\psi_i^{cond\to N}\right\rangle, \left|\psi_j^{cond\to N}\right\rangle$ obtained after the measurement of N photons in mode d is

$$\left\langle \psi_i^N \middle| \psi_j^N \right\rangle = \sum_{n,m} \langle 0| \tilde{Q}_{\psi_i}^{(N-m)\dagger} \tilde{Q}_a^{(m)\dagger} \tilde{Q}_a^{(n)} \tilde{Q}_{\psi_i}^{(N-n)} |0\rangle \tag{A2}$$

with $max\{0, N-n_s\} \leq n, m \leq min\{n_a, N\}$.

Let's first consider the case $N = n_a + n_s = N_{max}$:

$$\begin{aligned} &\left\langle \psi_i^{N_{max}} \middle| \psi_j^{N_{max}} \right\rangle = \\ &= \langle 0| \tilde{Q}_{\psi_i}^{(n_s)\dagger} \tilde{Q}_a^{(n_a)\dagger} \tilde{Q}_a^{(n_a)} \tilde{Q}_{\psi_j}^{(n_s)} |0\rangle \\ &= \langle 0| \tilde{Q}_a^{(n_a)\dagger} \tilde{Q}_a^{(n_a)} \tilde{Q}_{\psi_i}^{(n_s)\dagger} \tilde{Q}_{\psi_j}^{(n_s)} |0\rangle \\ &= \sum_{\{\mathbf{n}\}} \langle 0| \tilde{Q}_a^{(n_a)\dagger} \tilde{Q}_a^{(n_a)} |\mathbf{n}\rangle \langle \mathbf{n}| \tilde{Q}_{\psi_i}^{(n_s)\dagger} \tilde{Q}_{\psi_j}^{(n_s)} |0\rangle \\ &= \langle 0| \tilde{Q}_a^{(n_a)\dagger} \tilde{Q}_a^{(n_a)} |0\rangle \langle 0| \tilde{Q}_{\psi_i}^{(n_s)\dagger} \tilde{Q}_{\psi_j}^{(n_s)} |0\rangle \end{aligned} \tag{A3}$$

Above we have used the fact that $\left[\tilde{Q}_{\psi_i}^{(n_s)}, \tilde{Q}_a^{(n_a)\dagger}\right]=0$ [13] and introduced the completeness relation $\sum_{\{\mathbf{n}\}} |\mathbf{n}\rangle \langle \mathbf{n}|$, where $|\mathbf{n}\rangle$ is a Fock states of the relevant modes. Note that only the term corresponding to $|0\rangle\langle 0|$ survives.

Let's now evaluate (A2) when $N = N_{max} - 1$

$$\left\langle \psi_i^{N_{max}-1} \middle| \psi_j^{N_{max}-1} \right\rangle = \sum_{n,m=0}^{1} \mathcal{C}_{m,n}(i,j) \tag{A4}$$

where

$$\begin{aligned} &\mathcal{C}_{m,n}(i,j) = \\ &\quad = \langle 0| \tilde{Q}_{\psi_i}^{(n_s-m)\dagger} \tilde{Q}_a^{(n_a-1+m)\dagger} \tilde{Q}_a^{(n_a-1+n)} \tilde{Q}_{\psi_j}^{(n_s-n)} |0\rangle \end{aligned}$$

It is straightforward to verify [13] that $\left[\tilde{P}_{aux}, \tilde{P}_{\psi_i}^{\dagger}\right] = 0$ implies that

$$\left[\tilde{Q}_a^{(n_a)}, \tilde{Q}_{\psi_i}^{(n)\dagger}\right] = \left[\tilde{Q}_a^{(m)}, \tilde{Q}_{\psi_i}^{(n_s)\dagger}\right] = 0 \quad \forall\, m,n \tag{A5a}$$

and that:

$$\left[\tilde{Q}_a^{(n_a-1)}, \tilde{Q}_{\psi_i}^{(n_s-1)\dagger}\right] = n_a n_s \cdot \tilde{Q}_a^{(n_a)} \tilde{Q}_{\psi_i}^{(n_s)\dagger} \tag{A5b}$$

Relation (A5a), with a procedure analogous to the one used to derive (A3), can be used to show that

$$\begin{aligned} \mathcal{C}_{0,0}(i,j) &= \langle 0| \tilde{Q}_{\psi_i}^{(n_s)\dagger} \tilde{Q}_a^{(n_a-1)\dagger} \tilde{Q}_a^{(n_a-1)} \tilde{Q}_{\psi_j}^{(n_s)} |0\rangle \\ &= \langle 0| \tilde{Q}_a^{(n_a-1)\dagger} \tilde{Q}_a^{(n_a-1)} |0\rangle \langle 0| \tilde{Q}_{\psi_i}^{(n_s)\dagger} \tilde{Q}_{\psi_j}^{(n_s)} |0\rangle \end{aligned} \tag{A6}$$

Let's now consider terms

$$\begin{aligned} \mathcal{C}_{1,0}(i,j) &= \langle 0| \tilde{Q}_{\psi_i}^{(n_s-1)\dagger} \tilde{Q}_a^{(n_a)\dagger} \tilde{Q}_a^{(n_a-1)} \tilde{Q}_{\psi_j}^{(n_s)} |0\rangle \\ &= \left(\langle 0| \tilde{Q}_{\psi_j}^{(n_s)\dagger} \tilde{Q}_a^{(n_a-1)\dagger} \tilde{Q}_a^{(n_a)} \tilde{Q}_{\psi_i}^{(n_s-1)} |0\rangle \right)^* \\ &= \mathcal{C}_{0,1}^*(j,i) \end{aligned} \tag{A7}$$

which, with the help of (A5a) can be expressed as

$$\begin{aligned} \mathcal{C}_{1,0}(i,j) &= \langle 0| \tilde{Q}_a^{(n_a)\dagger} \tilde{Q}_a^{(n_a-1)} \tilde{Q}_{\psi_i}^{(n_s-1)\dagger} \tilde{Q}_{\psi_j}^{(n_s)} |0\rangle \\ &- \langle 0| \tilde{Q}_a^{(n_a)\dagger} \left[\tilde{Q}_a^{(n_a-1)}, \tilde{Q}_{\psi_i}^{(n_s-1)\dagger}\right] \tilde{Q}_{\psi_j}^{(n_s)} |0\rangle \end{aligned} \tag{A8}$$

As all the states ψ_i contain a definite number of photons, namely $\mathcal{N} = 2$, $\tilde{P}_{\psi_i}(d^\dagger, e_k^\dagger)$ is a homogeneous polynomial of degree $\mathcal{N}$ in $d^\dagger$ and $e_k^\dagger$ and therefore the generic $\tilde{Q}_{\psi_i}^{(n)}$ is a homogeneous polynomial of degree $\mathcal{N} - n$ in $e_k^\dagger$. As a consequence $\tilde{Q}_{\psi_i}^{(n_s-1)\dagger} \tilde{Q}_{\psi_j}^{(n_s)} |0\rangle = 0$. From this follows that the first term at the right hand side of (A8) is equal to zero.

Finally, with the help of (A5b) and (A5a) we obtain

$$\begin{aligned} \mathcal{C}_{1,0}(i,j) &= -n_a n_s \langle 0| \tilde{Q}_a^{(n_a)\dagger} \tilde{Q}_a^{(n_a)} \tilde{Q}_{\psi_i}^{(n_s)\dagger} \tilde{Q}_{\psi_j}^{(n_s)} |0\rangle \\ &= -n_a n_s \langle 0| \tilde{Q}_a^{(n_a)\dagger} \tilde{Q}_a^{(n_a)} |0\rangle \langle 0| \tilde{Q}_{\psi_i}^{(n_s)\dagger} \tilde{Q}_{\psi_j}^{(n_s)} |0\rangle \\ &= \mathcal{C}_{0,1}^*(j,i) = \mathcal{C}_{0,1}(i,j) \end{aligned} \tag{A9}$$

where again we have made use of a completeness relation.

We are left with the term $\mathcal{C}_{1,1}(i,j)$ in the sum of eq.(A4), which can be simplified with the same procedure as in eq.(A3) to obtain

$$\begin{aligned} \mathcal{C}_{1,1}(i,j) &= \langle 0| \tilde{Q}_{\psi_i}^{(n_s-1)\dagger} \tilde{Q}_a^{(n_a)\dagger} \tilde{Q}_a^{(n_a)} \tilde{Q}_{\psi_j}^{(n_s-1)} |0\rangle \\ &= \langle 0| \tilde{Q}_a^{(n_a)\dagger} \tilde{Q}_a^{(n_a)} |0\rangle \langle 0| \tilde{Q}_{\psi_i}^{(n_s-1)\dagger} \tilde{Q}_{\psi_j}^{(n_s-1)} |0\rangle \end{aligned} \tag{A10}$$

By inserting (A6, A9, A10) into (A4) we obtain

$$\begin{aligned} \left\langle \psi_i^{N_{max}-1} \middle| \psi_j^{N_{max}-1} \right\rangle &= \mathcal{A}_{n_s} \langle 0| \tilde{Q}_{\psi_i}^{(n_s)\dagger} \tilde{Q}_{\psi_j}^{(n_s)} |0\rangle \\ &+ \mathcal{A}_{n_s-1} \langle 0| \tilde{Q}_{\psi_i}^{(n_s-1)\dagger} \tilde{Q}_{\psi_j}^{(n_s-1)} |0\rangle \end{aligned} \tag{A11}$$

where, up to irrelevant multiplicative constants,

$$\begin{aligned} \mathcal{A}_{n_s} &= \langle 0| \tilde{Q}_a^{(n_a-1)\dagger} \tilde{Q}_a^{(n_a-1)} |0\rangle - \\ &- 2n_a n_s \langle 0| \tilde{Q}_a^{(n_a)\dagger} \tilde{Q}_a^{(n_a)} |0\rangle \\ \mathcal{A}_{n_s-1} &= \langle 0| \tilde{Q}_a^{(n_a)\dagger} \tilde{Q}_a^{(n_a)} |0\rangle \end{aligned} \tag{A12}$$

which concludes our proof, as, from (A3) and (A11) follows the implication between (A1a) and (A1b).

APPENDIX B

In this appendix we will prove that condition (21) is incompatible with (20) for both $i_0 = 0$ and $i_0 = 1$. To this goal it is helpful to define a matrix $\mathbf{M}$, linear combination of the $\mathbf{M}^{(i)}$:

$$\mathbf{M} = \sum_i \mu_i \mathbf{M}^{(i)}$$

so that a generic input state can be defined as

$$|\psi\rangle = \sum_i \mu_i |\psi_i\rangle = \mathbf{A}^T \mathbf{M} \mathbf{A} |0\rangle$$

From (2) follows that

$$\mathbf{M} = \frac{1}{\sqrt{2^3}} \begin{pmatrix} 0 & 0 & 0 & \mu_{-1}+\mu_1 & \mu_{-1}-\mu_1 & \mu_{-4}+\mu_4 & \cdots & 0 \\ 0 & 0 & 0 & \mu_{-3}+\mu_3 & \sqrt{2}\mu_1 & \mu_{-4}-\mu_4 & \cdots & 0 \\ 0 & 0 & 0 & \mu_{-3}-\mu_3 & \mu_{-2}+\mu_2 & \mu_{-2}-\mu_2 & \cdots & 0 \\ \mu_{-1}-\mu_1 & \mu_{-3}+\mu_3 & \mu_{-3}-\mu_3 & 0 & 0 & 0 & \cdots & 0 \\ \mu_{-1}-\mu_1 & \sqrt{2}\mu_1 & \mu_{-2}+\mu_2 & 0 & 0 & 0 & \cdots & 0 \\ \mu_{-4}+\mu_4 & \mu_{-4}-\mu_4 & \mu_{-2}-\mu_2 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

We can then write

$$\begin{aligned} \tilde{M}_{00} &= \underline{\mathrm{c}}_{\mathbf{0}}^{\mathbf{T}} \cdot \mathbf{M} \cdot \underline{\mathrm{c}}_{\mathbf{0}} \\ &= \mu_0 \cdot u_1 u_4 + \\ &+ \{\mu_1 \cdot u_0(u_3+u_4) + \mu_{-1} \cdot u_0(u_3-u_4) + \\ &+ \mu_2 \cdot u_2(u_4+u_5)/ + \mu_{-2} \cdot u_2(u_4-u_5) + \\ &+ \mu_3 \cdot u_3(u_1+u_2)/ + \mu_{-3} \cdot u_3(u_1-u_2) + \\ &+ \mu_4 \cdot u_5(u_0+u_1)/ + \mu_{-3} \cdot u_5(u_0-u_1)\}/\sqrt{2} \end{aligned} \tag{B1}$$

We now impose condition (21) with $i_0 = 0$ on vector $\underline{\mathrm{c}}_{\mathbf{0}}$, i.e. we impose that $\tilde{M}_{00}^{(0)}$ is the only nonzero coefficient. We have therefore to equal to zero all coefficients in (B1) except the one multiplying μ_0. The only solution compatible with this condition is

$$\underline{\mathrm{c}}_{\mathbf{0}} = (0, u_1, 0, 0, u_4, 0, \ldots, 0)^T \tag{B2}$$

From the form of $\mathbf{M}$ and from (B2) follows that (19) can be rewritten as follows:

$$\begin{aligned} \vec{\boldsymbol{M}}_0 &= \sum_i \mu_i \vec{\boldsymbol{M}}_0^{(i)} \\ &= \mu_0 \cdot \frac{u_1 \underline{\mathrm{r}}_4 + \underline{\mathrm{r}}_1 u_4}{2} + \frac{\mu_1 - \mu_{-1}}{2^{\frac{3}{2}}} u_4 \underline{\mathrm{r}}_0 + \\ &+ \frac{\mu_2 + \mu_{-2}}{2^{\frac{3}{2}}} u_4 \underline{\mathrm{r}}_2 + \frac{\mu_3 + \mu_{-3}}{2^{\frac{3}{2}}} u_1 \underline{\mathrm{r}}_3 + \frac{\mu_4 - \mu_{-4}}{2^{\frac{3}{2}}} u_1 \underline{\mathrm{r}}_5 \end{aligned} \tag{B3}$$

Condition (20) implies

$$\left\langle \psi_1^{cond\to 1} \middle| \psi_{-1}^{cond\to 1} \right\rangle \propto |u_4|^2 \|\underline{\mathrm{r}}_0\|^2 = 0 \tag{B4a}$$

$$\left\langle \psi_3^{cond\to 1} \middle| \psi_{-3}^{cond\to 1} \right\rangle \propto |u_1|^2 \|\underline{\mathrm{r}}_3\|^2 = 0 \tag{B4b}$$

Since condition (15) requires that $\|\underline{\mathrm{r}}_0\|^2 = \|\underline{\mathrm{r}}_3\|^2 = 1$, to fulfill (B4) we must impose $|u_1| = |u_4| = 0$. This however would imply $n_s = 0$.

We will now show that conditions (20), (21) cannot be simultaneously fulfilled with $i_0 = 1$, i.e that the only non zero coefficient of $\tilde{M}_{00}$ is the one multiplying the coefficient μ_1.

Along the same lines of the previous case we obtain a constrain on the vector $\underline{\mathrm{c}}_{\mathbf{0}}$ leading to the following relation

$$\underline{\mathrm{c}}_{\mathbf{0}} = (u_0, 0, 0, u, u, 0, \ldots, 0)^T$$

and $\vec{\boldsymbol{M}}_0$ reduces to

$$\begin{aligned} \vec{\boldsymbol{M}}_0 &= 2^{-\frac{2}{3}} \{\mu_1[u_0(\underline{\mathrm{r}}_3 + \underline{\mathrm{r}}_4) + \underline{\mathrm{r}}_1 2u] + \\ &+ \mu_{-1} u_0(\underline{\mathrm{r}}_3 - \underline{\mathrm{r}}_4) + (\mu_2 - \mu_{-2}) u \underline{\mathrm{r}}_2 + \\ &+ (\mu_4 + \mu_{-4}) u_0 \underline{\mathrm{r}}_5 + \mu_3 u(\underline{\mathrm{r}}_1 + \underline{\mathrm{r}}_2) + \\ &+ \mu_{-3} u(\underline{\mathrm{r}}_1 - \underline{\mathrm{r}}_2)\} \end{aligned} \tag{B5}$$

Therefore we obtain

$$\left\langle \psi_2^{cond\to 1} \middle| \psi_{-2}^{cond\to 1} \right\rangle \propto |u|^2 \|\underline{\mathrm{r}}_2\|^2 = 0$$

$$\left\langle \psi_4^{cond\to 1} \middle| \psi_{-4}^{cond\to 1} \right\rangle \propto |u_0|^2 \|\underline{\mathrm{r}}_5\|^2 = 0$$

From the unitarity condition (15) follows that $\|\underline{\mathrm{r}}_2\| = \|\underline{\mathrm{r}}_5\| = 1$ and (12a) can be satisfied only if u and u_0 are both zero.

As before this requirement leads to the trivial solution $\underline{\mathrm{c}}_{\mathbf{0}} = \underline{0}$, which is incompatible with $n_s > 0$

Both with $i_0 = 1$ and $i_0 = 0$ we obtain that conditions (20) and (21) lead to the trivial solution $\underline{\mathrm{c}}_{\mathbf{0}} = \underline{0}$, i.e. $n_s = 0$.

A fortiori conditions (20) and (17) will admit as only solution the trivial one. This implies indistinguishability also in the case $n_s = 1$.

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[13] To demonstrate these commutation relations write $\left[\tilde{P}_{aux}, \tilde{P}^{\dagger}_{\psi_i}\right] = 0$ as a power series of normally ordered $d^{\dagger}$, d operators, and impose that each term of this power series must vanish. The coefficients with power n_a, $n_a - 1$, n_s, and $n_s - 1$ include the commutators we want to calculate.